THE EFFECT OF A SOLID WALL ON THE CLOSURE OF A SPHERICAL CAVITATION POCKET

Yu. L. Levkovskii and G. G. Sudakova

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We examine the effect of a solid wall on the closure of a spherical cavitation cavity or pocket. It is demonstrated that asymmetric flow substantially reduces the induced pressure and rate of closure for the cavity containing the gas, while in the case of a vapor cavity asymmetry of flow leads to conclusions qualitatively different from those which follow from the classical Rayleigh solution for an infinite fluid.

Based on the method of reflections, the mathematical apparatus for the description of the phenomena involved in the closure of spherical cavities containing a gas near a solid boundary has been developed in detail [1]. The radial oscillations at the wall of a spherical cavity containing a large quantity of gas have been studied in [2]. We employ the same methods below to examine the closure at the wall of a cavity with a relatively low gas content, as is characteristic for the phenomenon of vapor cavitation.

Limiting ourselves to the reflection of a single source and a single dipole, which is equivalent to neglecting powers of $\varepsilon = R/2b$ above the first as small in comparison with unity, we can write the flow potential in the form

$$\varphi = R^2 \frac{dR}{dt}\left\{\frac{1}{(x^2+y^2)^{1/2}}+\frac{1}{[(x-2b)^2+y^2]^{1/2}}\right\} - \frac{R^3}{2}\frac{db}{dt}\left\{\frac{x}{(x^2+y^2)^{3/2}}-\frac{x-2b}{[(x-2b)^2+y^2]^{3/2}}\right\}.$$

The first terms in the braces correspond to the radial and translational motion of a real sphere, while the second terms correspond to the reflection of that motion at the point with the abscissa x = 2b.

The kinetic energy of the liquid can be calculated from the values of the potential and its derivatives at the boundary surfaces, these values having been determined from the boundary conditions; equally accurate is the following value of the kinetic energy:

$$T = 2\pi\rho R^3\left(\frac{dR}{dt}\right)^2(1+\varepsilon)+\frac{\pi\rho R^3}{3}\left(\frac{db}{dt}\right)^2. \quad (2)$$

There is no term which takes into consideration the mutual effect of the radial and translational motion on the magnitude of the kinetic energy, since that term has the order $O(\varepsilon^2)$. The second term in (2) represents the kinetic energy for the motion of a sphere in an infinite fluid, since the influence of the wall introduces a connection into the order, i.e., $O(\varepsilon^2)$. Use of the value for the kinetic energy from (2) by means of the method of generalized coordinates enables us to derive a system of differential equations for the radial and translational motion of the cavity:

$$\eta\ddot{\eta}(1+\varepsilon)+\dot{\eta}^2\left(\frac{3}{2}+2\varepsilon\right) - \frac{\dot{\beta}^2}{4}-2\varepsilon^2\dot{\eta}\dot{\beta}-\frac{\delta}{\eta^4}+1=0, \quad (3)$$

$$\eta\ddot{\beta}+3\dot{\eta}\dot{\beta}+6\varepsilon^2\dot{\eta}^2=0, \quad (3')$$

where

$$\eta=\frac{R}{R_0},\quad \beta=\frac{b}{R_0},\quad \dot{\eta}=\frac{dR}{dt}\left(\frac{\rho}{p_0}\right)^{1/2},$$

$$\dot{\beta}=\frac{db}{dt}\left(\frac{\rho}{p_0}\right)^{1/2},\quad \ddot{\eta}=\frac{d^2R}{dt^2}\frac{R_0\rho}{p_0},$$

$$\ddot{\beta}=\frac{d^2b}{dt^2}\frac{R_0\rho}{p_0},\quad \delta=\frac{p_{g0}}{p_0}.$$

The system of differential equations (3) has been derived with the assumption that the gas in the pocket is adiabatically compressed, i.e., $\gamma = 4/3$. With the cavity removed to infinity from the wall we have $\varepsilon \to 0$, $\dot{\beta} \to 0$, and $\ddot{\beta} \to 0$, and consequently, the left-hand part of the second equation is identically zero, while the first equation changes into the familiar equation for the radial motion of a cavity in an infinite fluid

$$\eta\ddot{\eta}+\frac{3}{2}\dot{\eta}^2-\frac{\delta}{\eta^4}+1=0. \quad (4)$$

The solution of this equation can be derived analytically [3]. The velocity of the pocket boundary is given by the relationship

$$\dot{\eta}^*=\left\{\frac{2}{3}\left[\eta^{-3}-1-4\delta(\eta^{-4}-\eta^{-3})\right]\right\}^{1/2}. \quad (5)$$

The maximum velocity [sic] on closure has the value

$$\dot{\eta}^*_{\max}=\left[\frac{(1+3\delta)^4-256\delta^3}{384\delta^3}\right]^{1/2} \approx 5.1\cdot10^{-2}(1+12\delta)^{1/2}\delta^{-3/2}. \quad (6)$$

The approximate equality here and below corresponds to low gas contents, i.e., $\delta \le 10^{-2}$. The minimum cavity radius is

$$\eta^*_{\min}=\frac{3\delta}{1+3\delta-\delta^{3/2}}\approx\frac{3\delta}{1+3\delta}. \quad (7)$$

The maximum pressure corresponds to the instant of closure, and when the condition $\varepsilon \le 1$ is satisfied we have

$$\vartheta^*_{\max} = \frac{p_{\max} - p_0}{p_0} = \\ = \frac{2\varepsilon(1+3\delta)(1+3\delta-\delta^{3/2})^2}{27\eta\delta^2} \approx \frac{2\varepsilon(1+9\delta)}{27\eta\delta^2}. \quad (8)$$

In the case of a vapor cavity ($\delta = 0$) we can calculate [4] the time of its complete closure:

$$\Delta\tau^* = \frac{\Delta t^*}{R_0}\left(\frac{p_0}{\rho}\right)^{1/2} = 0.915. \quad (9)$$

In analogy with the method employed in the solution of Eq. (4), excluding the time t and treating the radius η as an independent variable, we can reduce the order of system (3) to unity:

$$v'v\eta(1+\varepsilon) + v^2\left(\frac{3}{2} + 2\varepsilon - \frac{\beta'^2}{4} - 2\varepsilon^2\beta'\right) \\ \frac{\delta}{\eta^4} + 1 = 0, \quad (10)$$

$$\beta''\eta + \beta'\left(3 + \frac{v'\eta}{v}\right) + 6\varepsilon^2 = 0, \quad (10')$$

where

$$v = \dot{\eta}, \quad v' = \frac{dv}{d\eta}, \quad \beta' = \frac{d\beta}{d\eta}, \quad \beta'' = \frac{d^2\beta}{d\eta^2}.$$

Analytically, the systems of differential equations (3) and (10) cannot be integrated. The solution is achieved numerically, with the use of computers; system (3) was solved by the Runge-Kutta method, and system (10) was solved by the iteration method. The difference in the results of the solutions does not exceed 1%. Simultaneous with the determination of the kinematic characteristics of cavity motion, we calculated the pressure at the wall and at the critical point b, where the pressure is at its maximum. The utilization of the Lagrange-Cauchy integral and the value of the potential from (1) make it possible for us to derive an expression for the pressure at this point in the form

$$\vartheta = \frac{p - p_0}{p_0} \\ = 4\varepsilon(2\dot{\eta}^2 + \eta\ddot{\eta}) - 4\varepsilon^2(5\dot{\eta}\dot{\beta} + \eta\ddot{\beta}) + 16\varepsilon^3\beta^2. \quad (11)$$

In the case of an infinite fluid [3] at a great distance from the cavity

$$\vartheta^* = 2\varepsilon(2\dot{\eta}^2 + \eta\ddot{\eta}). \quad (12)$$

Consequently, with sufficiently adequate distance between the cavity and the wall—in which case the second and third terms in (11) can be neglected—the presence of the wall under the condition of equality between the radial velocities and accelerations leads to a doubling of the pressure.

The initial conditions for which the solution of the system of equations (3) and (10) was carried out corresponds to the assumption that the cavity is at rest at the initial instant of time and that the pressure difference between infinity and the inside of the cavity is given by $\Delta p = p_0(1 - \delta)$ when $t = 0$, $\eta = 1$, $\beta = \beta_0$, $\dot{\eta} = 0$, $\dot{\beta} = 0$. The cavity growth phase which follows the compression phase subsequent to the cavity reaching its minimum dimension $\eta_{\min}$ was not calculated, since the solution—as in the case of an infinite fluid—for $\eta_{\min}$ is symmetric.

The numerical calculations were carried out for four values of the gas contents: $\delta = 0$, 10^{-4}, 10^{-3}, 10^{-2}, and for seven values of the initial distance from the wall: $\beta_0 = 1.1$, 1.2, 1.5, 2.0, 5.0, 10, 100. For the case in which $\beta_0 \to \infty$, we use an analytical solution.

Figure 1 as an example shows the results from the calculation of cavity closure for the case $\beta_0 = 1.5$, $\delta = 10^{-4}$ in the form of curves showing $\dot{\eta}$, $\dot{\beta}$, $\ddot{\eta}$, $\ddot{\beta}$, $\chi = \beta/\beta_0$, $\alpha = \tau/\Delta\tau^*$, and ϑ as a function of the relative radius η. Analysis of these results shows that on closure the velocity of the cavity toward the wall is of the same order of magnitude as the radial velocity. The basic translational displacement of the cavity occurs during the final stages of closure, and when $\eta = \eta_{\min}$ the velocity of the translational motion reaches its maximum. The picture for the radial motion corresponds qualitatively to the closure of a gas-filled cavity in an infinite fluid; however, quantitatively speaking, the velocity, acceleration, and induced pressure are considerably smaller, while the minimum radius and closure time are greater. Figure 2 shows the results from a calculation of the velocity as a function of both the radius and the initial distance from the wall for a gas content of $\delta = 0$. Our attention is drawn to the fact that, qualitatively, the effect of a reduction in the initial distance from the wall is analogous to the effect of an increase in the gas content when the cavity in an infinite fluid is closed. The results from the calculations of the minimum cavity radius $\eta_{\min}(\beta_0; \delta)$ are shown in Fig. 3.

Analysis of the derived results shows that the greater the initial distances, the smaller the gas content and that these distances are affected by the pres-

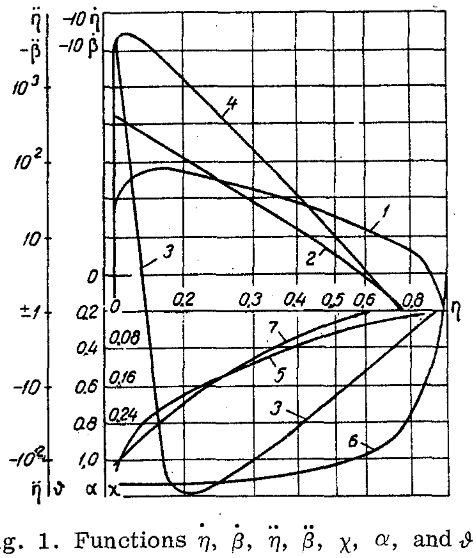

Fig. 1. Functions $\dot{\eta}$, $\dot{\beta}$, $\ddot{\eta}$, $\ddot{\beta}$, χ, α, and ϑ versus the relative radius η for $\beta_0 = 1.5$ and $\delta = 10^{-4}$: 1) $\dot{\eta}$; 2) β; 3) $\ddot{\eta}$; 4) $\ddot{\beta}$; 5) χ; 6) α; 7) ϑ.

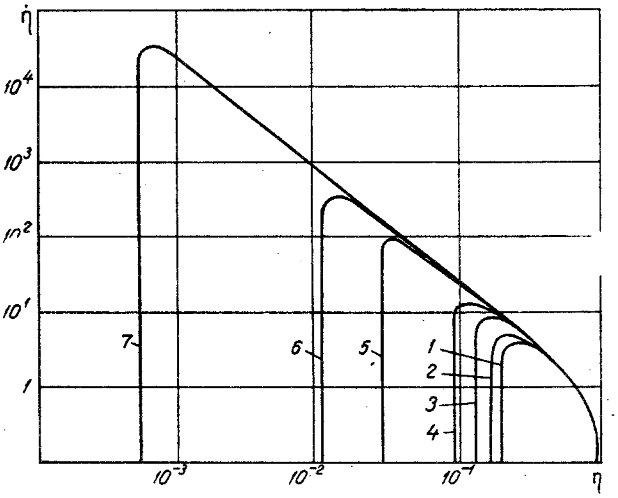

Fig. 2. Radial velocity of closing vapor cavity (δ = 0) as a function of the radius η for various initial distances from the wall: 1) β_0 = 1.1; 2) 1.2; 3) 1.5; 4) 2; 5) 5; 6) 10; 7) 100.

ence of the wall. Unanticipated is the substantial influence of the wall on the closure of a cavity with a gas content of $\delta \leq 10^{-4}$ at a distance as large as β_0 = 100. This result indicates the strong effect of even slight asymmetry of flow on the final stage of closure; this is also borne out by the qualitative difference in the behavior of the vapor cavity (δ = 0) at the wall and in an infinite fluid: despite the absence of gas in the cavity, complete closure of the cavity does not occur.

The explanation of the derived result lies in the fact that in the presence of a wall a portion of the potential energy—which the fluid exhibits at the initial instant of time—changes into the kinetic energy of translational motion, so that the presence of the wall leads to a reduction in the velocity of radial motion for the boundary of the cavity, while in the case of a vapor cavity it leads to the appearance of conditions under which its complete closure proves to be impossible. This conclusion is a consequence of the assumption that the cavity retains its spherical shape. Indeed, because of the accelerated translational motion, the diagram showing the pressure distribution over the surface of the cavity is exceedingly nonuniform, in connection with which—in the later stages of closure—the cavity must undergo strong deformation, subsequently collapsing. It can be demonstrated that the distribution of pressures over the surface of a moving sphere is described by the relation

$$\vartheta_s = \dot{\beta}^2 \frac{9\cos\theta - 5}{8} + \beta\ddot{\beta}\frac{\cos\theta}{2}, \tag{13}$$

where

$$\theta = \operatorname{arctg}\frac{y}{x}.$$

In connection with the condition that the cavity retain its spherical shape, in the over-all balance of forces applied to the boundary of the cavity, the surface-averaged pressure plays a role:

$$\bar{\vartheta}_s = \frac{1}{4\pi R^2}\int_s \vartheta_s\, ds = \dot{\beta}^2 \times$$

$$\times\left(\frac{9}{16}\int_0^\pi \cos^2\theta\sin\theta\, d\theta - \frac{5}{16}\int_0^\pi \sin\theta\, d\theta\right) +$$

$$+\frac{\beta\ddot{\beta}}{4}\int_0^\pi \cos\theta\sin\theta\, d\theta = -\frac{\dot{\beta}^2}{4}. \tag{14}$$

The last integral in (14) is equal to zero; consequently, the average pressure resulting from the nonsteady nature of the motion is equal to zero. Thus, because of the translational motion of the cavity, a pressure is developed on the cavity surface, the average magnitude of this pressure is determined by the velocity, and it is a negative value, which is equivalent to the presence in the cavity of a gas with a pressure determined from relationship (14). When the cavity attains its minimum dimension, its velocity of translational motion and, consequently, its tensile stresses are at their maximum.

Figure 4 shows the results from the calculation of the maximum pressure at the wall at the instant of cavity closure, referred to the pressure induced by

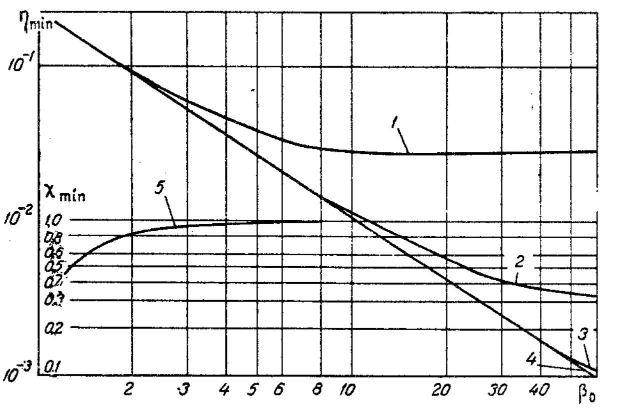

Fig. 3. Minimum cavity radius $\eta_{min}(\beta_0\delta)$ and function $\chi_{min}(\beta_0)$: 1) $\delta = 10^{-2}$; 2) 10^{-3}; 3) 10^{-4}; 4) 0; 5) $\chi_{min}(\beta_0)$.

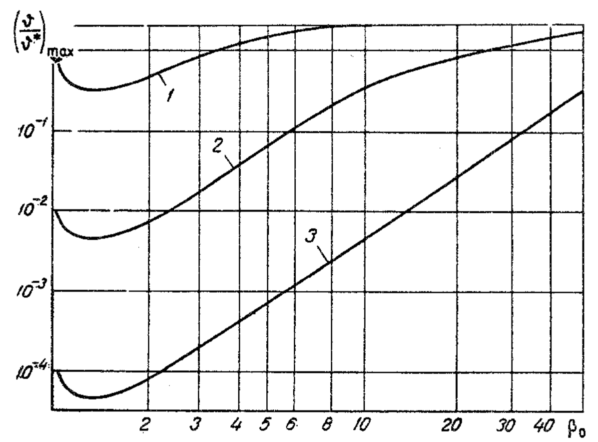

Fig. 4. Maximum pressure on wall at the instant of cavity closing as a function of β_0: 1) $\delta = 10^{-2}$; 2) 10^{-3}; 3) 10^{-4}.

the cavity on its closure in an infinite fluid at the same distance.

With a reduction in the initial distance from the wall, the ratio of the induced pressure to the pressure on cavity closure in an infinite fluid initially diminishes, reaching a minimum at $\beta_0 \approx 1.35$, and then increasing slowly, which is explained by the significant influence of the approach of the cavity to the wall during the closure process when $\beta_0 < 1.35$. Accurate to ~0.1%, the quantity χ_{min} is found to be independent of gas content δ; the curve of the function $\chi_{min}(\beta_0)$ is shown in Fig. 3. The general trend toward a reduction in the ϑ/ϑ^* ratio with a reduction in β_0 is obviously associated with a reduction in the radial acceleration of the cavity at the instant of closure. As we can see from the graph, for low gas contents the induced pressure may be many orders of magnitude smaller than in an infinite fluid and, consequently, the large values for the induced pressure predicted by theory [3, 4] at the instant of closure are in actual fact impossible.

The time for the complete closure of the cavity is greater than that calculated by Rayleigh [4]; however, it does not differ from that quantity significantly.

This investigation permits us to draw the important conclusion that attempts to determine the kinematic characteristics of cavity motion during the final stages of closure—involving the use of the assumption of spherical symmetry of flow—cannot yield satisfactory results. The asymmetry existing under real conditions is caused by the proximity of the boundaries or by other nonuniformities in the flow as, for example, adjacent cavities, and so it must lead to an excessively pronounced quantitative difference between the theoretical and actual parameters of cavity motion. The experimentally recorded and unexplained pronounced reduction relative to the theoretical value for the radial velocity of the cavity during the final closure stages [5] is in all probability precisely a result of this circumstance.

NOTATION

p, ρ, T, and φ are the pressure, density, kinetic energy, and potential of the fluid flow; $p_0 = p_\infty - p_s$ is the pressure difference at infinity and of the saturated vapors; ϑ is the dimensionless pressure; p_{g0} is the initial gas pressure in cavity; δ is the relative gas content; γ is the adiabatic exponent; R and b are the radius of spherical cavity and the distance from its center to the wall as a function of time; R_0 and b_0 are the same at the initial instant; γ is the relative change of cavity distance from wall; t is the time; $\tau = t/R_0(p_0/\rho)^{1/2}$ is the dimensionless time; $\Delta\tau^*$ is the time of complete closing of the vapor cavity in the infinite fluid; x and y are the Cartesian coordinates, origin at sphere center, with the x-axis directed to the wall.

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